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Equations of Motion of Flexible Spacecraft

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This paper presents the derivation of the equations governing the motion of a flexible spacecraft. The spacecraft is modeled as a rigid central body with an arbitrary number of subsidiary flexible bodies each separately pinned to the central body at an arbitrary point with no interconnections between the subsidiary bodies. Forcing functions caused by environmental forces and torques are included in the spacecraft model. The differential equations describe the perturbed motion due to flexibility about an arbitrary nominal motion obtained when the spacecraft is modeled as an interconnected system of rigid bodies. This approach leads to a system of time-varying linear ordinary differential equations in which the time-varying coefficients depend on the nominal motion of the rigid body system which is presumed to be available from another source. The flexible characteristics of the subsidiary bodies in this derivation are obtained from existing digital computer programs for structural dynamics analysis.

Nomenclature

$\bar{a}$ = acceleration of the main body origin of coordinates, the nominal acceleration of the main body
A_α = mass properties associated with the main body, $A_\alpha = \int x_\alpha^2 dm$ $\alpha = 1,2,3$
A, A^i = sunlit area of the main body and body i, respectively
$\bar{b}^i$ = position of attachment point of body i to the main body relative to main body origin
C_k^i = scalar defined in Eq. (24)
$\bar{d}^i$ = position of body i mass center relative to body i axes
D_1^i = dissipation function associated with body i deformation
D_2^i = the dissipation function corresponding to the discrete dampers located at the ith attachment point
D_{kj}^i = the kj element of the equivalent viscous damping matrix of body i
E^i = mass property dyad of body i defined in Eq. (8)
E_j = unit vectors fixed in the Earth inertial frame
$\bar{g}_\alpha^i$ = a vector providing the small rotations of the point at the origin of body i, $\bar{g}_\alpha^i = \bar{\nabla} \times \bar{\phi}_\alpha(0)$
$\bar{h}_\alpha^i$ = a vector providing the small rotations of the point on the main body at which body i attaches, $\bar{h}_\alpha^i = -\bar{\nabla} \times \bar{\phi}_\alpha(\bar{b}^i)$
$[I]$ = the identity matrix
$I_{\alpha\beta}$ = the principal moment of inertia tensor of the main body $I_{\alpha\alpha}$ may be expressed in terms of A_α, e.g., $I_{11} = A_2 + A_3$
$\tilde{J}(\bar{r})$ = an operator that generates a skew symmetric dyad from the vector $\bar{r}$,

$$\tilde{J}(\bar{r}) = \begin{vmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{vmatrix}$$

$\tilde{K}_s^i$ = a dyad representing the stiffness of the springs acting about the attachment point of body i to the main body
K_{kj}^i = generalized stiffness matrix for body i
$\bar{M}^i$ = nominal spring torque acting on body i at the attachment point to the main body
m, m^i = masses of main body and body i, respectively
m_{kj}^i = generalized mass matrix for body i
$\tilde{N}^i$ = a mass property dyad of body i defined in Eq. (8)
n_{fi} = designation of the number of rigid body degrees of freedom of body i relative to the main body, n_{fi} = 0, 1, 2, or 3
q_α, q_k^i = time-varying generalized coordinates associated with the main body rotations (α = 1,2,3) and for the kth mode associated with body i, respectively
Q_H, Q_g = solar and gravity generalized forces; $Q_{HT\beta}, Q_{gT\beta}$ are associated with the main body translational coordinates p_β; and $Q_{HR\beta}, Q_{gR\beta}$, with the main body rotational coordinates q_β; and Q_{Hk}^i, Q_{gk}^i with body i flexible coordinates q_k^i
$\bar{R}_k^i$ = mass property of the kth mode of body i defined in Eq. (8)
$\bar{r}, \bar{r}^i$ = position of a field point in the main body relative to the main body origin, and in body i relative to body i origin, respectively
$\bar{S}^i$ = position vector of a surface point in body i measured relative to body i attachment point
T, T^m, T^i = total, main body and body i kinetic energies, respectively
T_R, T_R^i = main body and body i kinetic energy if the system is rigid
U, U^i = total strain energy and body i strain energy caused by deformation, respectively
$\tilde{U}$ = unit dyad
$\bar{V}$ = velocity of the main body origin of coordinates, the nominal velocity of the main body

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$\bar{p}, \bar{p}^i$ = change in position of the main body and body i origins when the system of bodies is assumed flexible

$\bar{W}, \bar{W}^i$ = changes in position of a field point in the main body, and in body i, respectively, caused by flexibility of the system

$\bar{X}, \bar{X}^i$ = positions of the main body origin and of the body i origin in inertial space if the system is assumed composed of rigid bodies

$\hat{x}_\alpha, \hat{x}_\alpha{}^i$ = unit vectors fixed in the main body and at body i attachment point, respectively, while the system is moving with nominal motion

$\bar{Y}_k{}^i, \bar{Z}_{kj}{}^i$ = mass properties of body i defined by Eq. (8)

$\bar{\beta}^i$ = damping dyad defining the discrete dampers acting at the ith attachment point

γ = gravitational constant for the Earth

$\theta_\alpha{}^i$ = angles defining the position of body i relative to the main body when the system is assumed composed of rigid bodies, $\alpha = 1,2,3$

$\Delta\bar{\theta}_\alpha{}^i$ = change in $\theta_\alpha{}^i$ at the attachment point when the system is assumed flexible

$\bar{\rho}, \bar{\rho}^i$ = position of a field point in the main body and body i in inertial space when the system is assumed flexible

$\bar{\phi}_\alpha$ = rigid body modes associated with the main body when the system is assumed flexible, $\alpha = 1,2,3$

$\bar{\phi}_j{}^i$ = the jth mode associated with body i

$\bar{\omega}, \bar{\omega}^i$ = total angular velocity of the main body and body i when the system is composed of rigid bodies

$\bar{\nabla}$ = vector differential operator, $\bar{\nabla} = \frac{\partial}{\partial x_1}\hat{x}_1 + \frac{\partial}{\partial x_2}\hat{x}_2 + \frac{\partial}{\partial x_3}\hat{x}_3$

Subscripts

α, β = 1,2,3

Introduction

THE flexibility effects on vehicle dynamics have received much attention in the design of aircraft and guided missiles. Spacecraft dynamics differs from the dynamics of aircraft and missiles in the following important aspects: 1) Many spacecraft are much more flexible than aircraft or missiles and are subject to more severe weight limitations; 2) Spacecraft components often undergo large relative rotations, complicating the dynamics of the system; and 3) Spacecraft are often subject to stringent stability and pointing accuracy requirements.

Several analytical investigations of flexible spacecraft, although limited to very simple structures, have shown the profound effect that structural elasticity has on spacecraft stability. Reiter[1] demonstrated this conclusion for a gravity-oriented satellite modeled as a simple beam. Robe and Kane[2] reached the same conclusion for a dumbell-shaped rotating satellite and Robe[3] extended the analysis to the case of two tethered unsymmetrical Earth-pointing bodies. A NASA monograph[4] on the dynamic interaction of flexible space vehicles contains relevant case histories, references and guidance for the designer of controlled flexible spacecraft. Likins[5] reviewed and developed several approaches to the derivation of the equations of motion of such systems, which are assumed to include both rigid and flexible components. He covered three basic approaches to the simulation of flexible space vehicles: discrete coordinate methods, hybrid coordinate methods, and vehicle normal coordinate methods.

The discrete coordinates methods require the application of Newton's Laws to a collection of interconnected rigid bodies. They involve few restrictions or approximations in the derivation but are very limited in application to flexible spacecraft because of the difficulty of creating the required mathematical model of a real vehicle without exceeding the practical limits imposed on the computations by cost and computer capacity considerations.

The hybrid coordinates methods are applied[5] to spacecraft which may be idealized as a collection of one or more rigid bodies to each of which one or more flexible appendages are attached. These methods employ position and attitude coordinates for the rigid bodies of the system and modal deformation coordinates for the elastic appendages. Two methods of hybrid coordinates are proposed by Likins. In one, the equations of motion of the flexible appendages and the total vehicle are derived separately, assuming the appendages to undergo only "small" deformations relative to a reference frame which maintains an orientation relative to the supporting rigid body as prescribed by a direction cosine matrix C, which in general is time dependent. When the nominal inertial angular velocity of the rigid body is zero, transformation of the appendage equations into modal coordinates in the real domain§ can be affected resulting in the uncoupling of the flexible degrees of freedom.

In the second or "synthetic modes" method of hybrid coordinates, the equations of motion to be solved are those of the rigid primary structure which include the forces and torques imparted by the flexible appendages. To facilitate modal coordinate transformation in the real domain, it is assumed that the rigid body undergoes only small displacement in inertial space and that the relative angular rates of the flexible components with respect to the rigid body are small (although the angular displacement itself may be large). The main feature of the hybrid coordinate methods is the capability to uncouple and truncate the flexible coordinates while retaining discrete dampers and rigid rotors in the simulation. However, the conditions imposed by the modal transformation limit the application of the truncated equations to the simulation of the spacecraft steady-state motion.

Finally, the vehicle normal coordinates methods[5] involve transformation of all the kinematic coordinates of the simulation and not merely the appendage deformation coordinates. To facilitate modal transformation of the kinematic coordinates in the real domain, the spacecraft system must be nonrotating and void of discrete dampers, thus limiting greatly the usefulness of this approach.

This paper presents the formulation of the equations of motion of n flexible bodies, all attached to one rigid body and undergoing large relative rotations. The total motion of the system is described as a superposition of linear elastic deformation modes upon the arbitrary nominal large displacement of the structure obtained when it is modeled as an interconnected system of rigid bodies. The development, therefore, is completely general and can be used to simulate acquisition maneuvers as well as the steady-state motion of space vehicles.

Discussion

The spacecraft configuration chosen for analysis consists of a central rigid body and an arbitrary number of flexible subsidiary bodies, each having one attachment point to the central body with no interconnections between subsidiary bodies. Each subsidiary body has up to three rotational degrees of freedom relative to the main body, with springs and discrete dampers acting at the attachment point. Large-angle rotations of the central body and of each subsidiary body relative to the central body are allowed when the system is modeled as an interconnected set of rigid bodies. The flexible characteristics of each subsidiary body are expressed in terms of three-dimensional elastic deformation functions. These functions and the associated mass and stiffness matrices form part of the input to the simulation and are ob-

§ The discussion is limited here to transformations in the real domain because of the exorbitant cost of performing simulation calculations with both real and complex numbers as required for transformations employed in Ref. 5 when the vehicle is rotating at a constant (nonzero) velocity.

tained from an existing digital computer program for structural dynamics analysis. Energy is dissipated in the spacecraft system by discrete dampers acting at the attachment points and structural damping in the subsidiary bodies. The latter is expressed in terms of equivalent viscous damping of the flexible degrees of freedom including a capability to simulate dissipative coupling.

The approach that is described herein requires that the vehicle's motion be represented as a small perturbation of an arbitrary nominal motion obtained by treating the vehicle as a collection of rigid bodies. Thus, the rigid body modeling constitutes the first part of the proposed simulation, and the nominal solution is obtained with a digital computer program which solves the equations of motion of an interconnected set of rigid bodies derived by the discrete coordinates method.[5] Several versions of this program now exist at various NASA facilities and aerospace companies. With the nominal motion available, the equations governing the perturbed motion are time varying linear ordinary differential equations. The time-varying coefficients depend on the nominal motion which may be discontinuous due to impulsive environmental and control forces and torques acting on the system. When the nominal motion is discontinuous, the perturbed dynamics and control equations are piecewise linear in time.¶

The perturbation approach to the simulation of a flexible space vehicle subject to a linear control system offers several important advantages when compared to a unified approach where a system of nonlinear ordinary differential equations is used to describe the total motion of the vehicle. The latter approach is obtained if the limitations on the reference frame motion imposed by the modal transformation requirements in the hybrid coordinates method of Ref. 5 are relaxed. The main advantages of the perturbation approach are:

1) Relevant information concerning the stability may be obtained from the linear perturbation equations prior to their integration. For slowly varying time dependent coefficients, an indication of local stability may be established by the algebraic solution of the associated eigenvalue problem at selected times. This information may be used to determine critical times in orbit most likely to reveal spacecraft instability and thus reduce considerably the total simulation. Stability analyses based on Floquet methods are not applicable (except in special cases) since the time varying coefficient matrices of the coupled linear equations of motion are, in general, nonperiodic. In contrast, the stability of the nonlinear differential equations in the unified approach may be ascertained by detailed and costly integration over a long period of simulated time.

2) The relatively inexpensive linear stability analysis of the perturbation approach may be used to judge the number of flexible modes needed to describe the flexible response adequately. With the unified approach this may be done only by observing the variation in the solution of the differential equations when it is repeated several times with a different number of flexible modes participating in the solution each time.

3) The ability to analyze the effects of flexibility on the spacecraft motion by perturbation of an analytically prescribed nominal motion constitutes a distinct advantage in fast, low-budget preliminary design efforts.

In a practical sense, the inherent assumption of "small" deviations from the nominal motion does not limit the usefulness of this approach. Most next-generation spacecraft require fine pointing accuracy, and any spacecraft system for which the perturbed motion due to flexibility is large would presumably be redesigned.

¶ An exception to this statement is the case where the nominal state vector follows a singular trajectory as defined in Ref. 6. Singular rigid body trajectories, however, may be regarded as pathological and normally would not be subjected to perturbation.

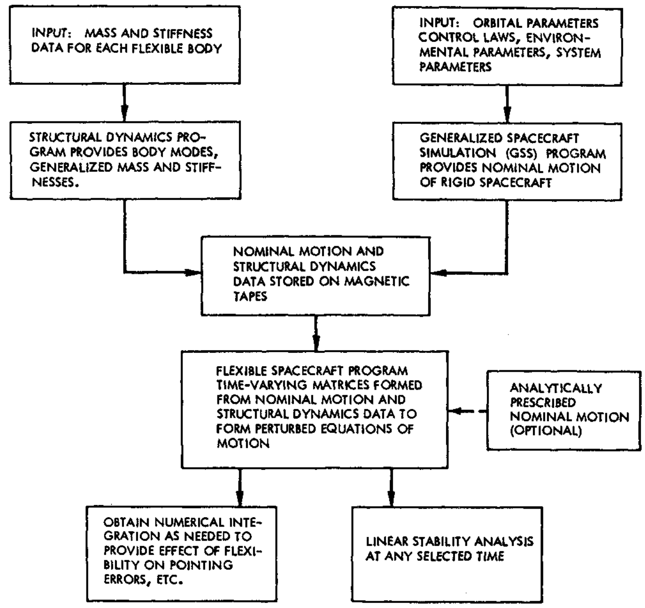

Fig. 1 Major steps to obtain flexible simulation.

Figure 1 presents a flow chart showing the principal steps in this approach, and Fig. 2 shows how it is envisaged that the program based on the formulation would be used in the design process. The design is first verified in the usual manner using a rigid body program. This condition satisfied, the structural dynamic inputs are established using a structural dynamic program. Before determining detailed motion for the flexible system, a linear stability analysis is performed to establish points in time at which the motion may diverge from the nominal and to determine the number of modes needed in the detailed integration. More detailed analyses are performed at these times to verify the acceptability of the motion of the system.

Basic Formulation

A Lagrangian approach is taken to formulate the equations of motion.

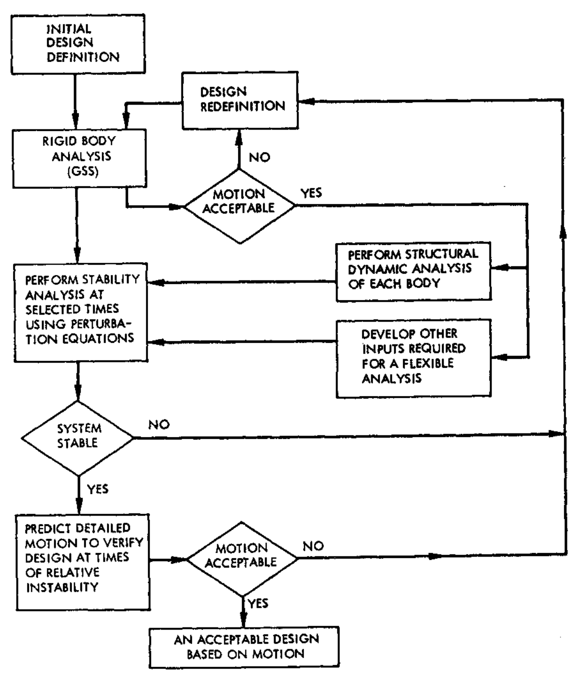

Fig. 2 Program use in design process.

Figure 3 presents a spacecraft in the nominal and perturbed position and shows vectors used to define the configuration. The main body origin 0 is located at the main body mass center and body i origin at body i attachment point. The nominal positions of the main body and body i origins in inertial space are given by vectors $\bar{X}$, $\bar{X}^i$, respectively.

Field points in the main body and body i relative to their origins are given by vectors $\bar{r}$ and $\bar{r}^i$, respectively. $\bar{b}^i$ locates body i attachment point relative to point 0. Vectors $\bar{\rho}$ $\bar{\rho}^i$ define the location of field points in the perturbed position and $\bar{p}$, $\bar{p}^i$ locate origins in the perturbed position of the system relative to their nominal positions. Finally, $\bar{W}$ defines the perturbed main body displacement field caused by small main body rotations and $\bar{W}^i$ defines the deformed state of body i relative to its underformed state. In the analysis, $\bar{W}^i$ is assumed to be representable as a summation of vector field functions with time varying coefficients.

The equations of motion of the system are derived by use of Lagrange's equation which is

$$(d/dt)(\partial T/\partial \dot{q}_i) - \partial T/\partial q_i + \partial D/\partial \dot{q}_i + \partial U/\partial q_i = Q_i \quad (1)$$
$$i = 1,2, \ldots$$

where T is the kinetic energy, D the dissipation function, U the strain energy and Q_i the generalized force corresponding to the coordinate q_i. In the present case, the main body has six degrees of freedom and each subsidiary body is given n_i degrees of freedom with respect to the main body; a total of

$$6 + \sum_{i=1}^{n} n_i$$

equations of motion are obtained. The n_i degrees of freedom of body i consist of n_{f_i} ($n_{f_i} = 0,1,2,3$) rigid body modes, depending on the degrees of freedom in the hinge point, and $n_i - n_{f_i}$ flexible degrees of freedom.

The generalized coordinates chosen to describe the perturbed configuration are three main-body translation coordinates, p_α three main-body rotations q_α, and coordinates for body, i, q_k^i. It is convenient to introduce three coordinates p_α^i to define the position of body i origin relative to its nominal position. The p_α^i can be written in terms of p_α, q_α and, therefore, are not independent coordinates and must either be eliminated from T and Q_i where they appear explicitly, or the constrained form of Lagrange's equation must be used. The latter approach is taken. In the remainder of this section, we derive expressions for T, D, U, and Q_i.

System Kinetic Energy

The system kinetic energy is the sum of the kinetic energies of the individual bodies. Thus, it is only necessary to derive the kinetic energy for an arbitrary body and then sum over the bodies to obtain the total energy. Referring to Fig. 3, the position of a mass element in body i in the perturbed configuration is

$$\bar{\rho}^i = \bar{X}^i + \bar{p}^i + \bar{r}^i + \bar{W}^i \quad (2)$$

and $\bar{W}^i$ is represented in the form

$$\bar{W}^i = \sum_{k=1} \bar{\phi}_k^i q_k^i \quad (3)$$

Since each subsididary body is assumed pinned to the main body with rotational springs and dashpots acting about the attachment point, any virtual slope change at the attachment point results in virtual work performed by the springs and dashpots. To minimize the number of terms of this type which may arise, the deformation functions are derived by assuming body i to be fixed at the attachment point. That is, the coordinates q_k^i, introduced in Eq. (3), are not defined by any transformation which diagonalizes Eqs. (1) in this paper, and indeed, for a general nominal motion, no such transformation exists. The coordinates q_k^i are the modal coordinates for vibration of the appendage when cantilevered at the hinge point from a fixed base. Thus, all slope changes at the attachment point are expressed in terms of the rigid body modes.

The first n_{f_i} modes ($n_{f_i} = 0, 1, 2,$ or 3) are the rigid body modes. These are defined as follows for $n_{f_i} = 3$:

$$\bar{\phi}_2^i = x_3^i\hat{x}_1^i - x_1^i\hat{x}_3^i \qquad \bar{\phi}_1^i = x_2^i\hat{x}_3^i - x_3^i\hat{x}_2^i$$
$$\bar{\phi}_3^i = x_1^i\hat{x}_2^i - x_2^i\hat{x}_1^i \quad (4)$$

The attachment point of body i is at the origin of the coordinate frame fixed in body i. Since, as mentioned previously, the flexible modes are derived by assuming body i to be fixed at its attachment point to the main body, the flexible modes of body i satisfy the conditions

$$\left.\begin{aligned} \bar{\phi}_k^i(0) &= 0 \\ \bar{\nabla} \times \bar{\phi}_k^i(0) &= 0 \end{aligned}\right\} k > n_{f_i} \quad (5)$$

The velocity of the mass point is

$$d\bar{\rho}^i/dt = \dot{\bar{X}}^i + \dot{\bar{p}}^i + \bar{\omega}^i \times (\bar{r}^i + \bar{W}^i) + \dot{\bar{W}}^i \quad (6)$$

where $\bar{\omega}^i$ is the inertial angular velocity of body i. The vectors $\dot{\bar{X}}$ and $\dot{\bar{p}}^i$ are derivatives calculated in an inertial reference frame, while $\dot{\bar{W}}^i$ is calculated in a reference frame in which $\hat{x}_\alpha^i$ is fixed. The kinetic energy T of the system is then

$$T = T^m + \sum_{i=1}^{n} T^i = \frac{1}{2}\int_{B_m} \dot{\bar{\rho}}\cdot\dot{\bar{\rho}}dm + \frac{1}{2}\sum_{i=1}^{n}\int_{B_i} \dot{\bar{\rho}}^i\cdot\dot{\bar{\rho}}^i dm^i \quad (7)$$

where T^m is the main body kinetic energy and T^i is the kinetic energy of body i. The integrals extend over the main body B_m and body i,B_i. The kinetic energy expressions T^m,T^i are derived in Ref. 7; T^i is presented below and is composed of 15 terms.

$$\begin{aligned} T^i = {} & T_R^i + (m^i/2)(2\dot{\bar{X}}^i\cdot\dot{\bar{p}}^i + \dot{\bar{p}}^i\cdot\dot{\bar{p}}^i) + \\ & \dot{\bar{X}}^i\cdot\Big(\bar{\omega}^i \times \sum_k q_k^i\bar{R}_k^i\Big) + \dot{\bar{X}}^i\cdot\sum_k \dot{q}_k^i\bar{R}_k^i + \\ & \dot{\bar{p}}^i\cdot(\bar{\omega}^i \times \bar{d}^i)m^i + \dot{\bar{p}}^i\cdot\Big(\bar{\omega}^i \times \sum_k q_k^i\bar{R}_k^i\Big) + \\ & \dot{\bar{p}}^i\cdot\sum_k \dot{q}_k^i\bar{R}_k^i + \bar{\omega}^i\cdot\Big(\sum_k q_k^i\tilde{N}_k^i\Big)\cdot\bar{\omega}^i + \\ & \bar{\omega}^i\cdot\sum_k \dot{q}_k^i\bar{Y}_k^i + \tfrac{1}{2}\,\bar{\omega}^i\cdot\Big(\sum_k\sum_l q_k^i q_l^i\tilde{E}_{kl}^i\Big)\cdot\bar{\omega}^i + \\ & \bar{\omega}^i\cdot\sum_k\sum_j q_k^i\dot{q}_j^i\bar{Z}_{kj}^i + \tfrac{1}{2}\,(\dot{q}_k^i)^T[m_{kj}]^i(\dot{q}_j^i) \end{aligned} \quad (8)$$

where T_R^i is body i nominal kinetic energy. $\bar{d}^i$ is the vector defining body i mass center relative to body i origin and $[m_{kj}]^i$ is the usual generalized mass matrix. Vectors $\bar{R}_k^i$, $\bar{Y}_k^i$, $\bar{Z}_{kj}^i$ and dyads $\tilde{N}_k^i$, $\tilde{E}_{kj}^i$ arise in the formulation due to body i flexibility and are defined below

$$\bar{R}_k^i = \int_{B_i} \bar{\phi}_k^i dm^i \quad (9)$$

$$\tilde{N}_k^i = \int_{B_i} \times \begin{bmatrix} (r_2\phi_2 + r_3\phi_3)\hat{x}_1\hat{x}_1 & -r_1\phi_2\hat{x}_1\hat{x}_2 & -r_1\phi_3\hat{x}_1\hat{x}_3 \\ -r_2\phi_1\hat{x}_2\hat{x}_1 & (r_1\phi_1 + r_3\phi_3)\hat{x}_2\hat{x}_2 & -r_2\phi_3\hat{x}_2\hat{x}_3 \\ -r_3\phi_1\hat{x}_3\hat{x}_1 & -r_3\phi_2\hat{x}_3\hat{x}_2 & (r_1\phi_1 + r_2\phi_2)\hat{x}_3\hat{x}_3 \end{bmatrix}^i dm^i \quad (10)$$

$$\bar{Y}_k^i = (N_{32} - N_{23})_k{}^i\hat{x}_1^i + (N_{13} - N_{31})_k{}^i\hat{x}_2^i + (N_{21} - N_{12})_k{}^i\hat{x}_3^i \quad (11)$$

$$\tilde{E}_{kl}{}^{i} = \int_{B_i} \begin{bmatrix} [\phi_{2k}\phi_{2l} + \phi_{3k}\phi_{3l})\hat{x}_1\hat{x}_1 & -\phi_{1k}\phi_{2l}\hat{x}_1\hat{x}_2 & -\phi_{1k}\phi_{3l}\hat{x}_1\hat{x}_3 \\ -\phi_{1k}\phi_{2l}\hat{x}_2\hat{x}_1 & (\phi_{1k}\phi_{1l} + \phi_{3k}\phi_{3l})\hat{x}_2\hat{x}_2 & -\phi_{2k}\phi_{3l}\hat{x}_2\hat{x}_3 \\ -\phi_{1k}\phi_{3l}\hat{x}_3\hat{x}_1 & -\phi_{2k}\phi_{3l}\hat{x}_3\hat{x}_2 & (\phi_{1k}\phi_{1l} + \phi_{2k}\phi_{2l})\hat{x}_3\hat{x}_3 \end{bmatrix} dm^i \quad (12)$$

and,

$$\bar{Z}_{kj}{}^{i} = (E_{23} - E_{32})_{kj}{}^{i}\hat{x}_1{}^{i} + (E_{31} - E_{13})_{kj}{}^{i}\hat{x}_2{}^{i} + (E_{12} - E_{21})_{kj}{}^{i}\hat{x}_3{}^{i} \quad (13)$$

Similarly, the kinetic energy of the main body is found to be

$$T^m = T_R + (m/2)\,(\dot{\bar{p}}\cdot\dot{\bar{p}} + 2\dot{\bar{x}}\cdot\dot{\bar{p}}) + q_1\omega_2\omega_3(A_3 - A_2) + q_2\omega_1\omega_3(A_1 - A_3) + q_3\omega_1\omega_2(A_2 - A_1) + \dot{q}_1\omega_1(A_2 + A_3) + \dot{q}_2\omega_2(A_3 + A_1) + \dot{q}_3\omega_3(A_1 + A_2) - q_{1/2}{}^2(A_2\omega_3{}^2 + A_3\omega_2{}^2) - (q_2{}^2/2)(A_3\omega_1{}^2 + A_1\omega_3{}^2) - (q_3{}^2/2)(A_1\omega_2{}^2 + A_2\omega_1{}^2) + q_1q_2A_3\omega_1\omega_2 + q_2q_3A_1\omega_2\omega_3 + q_1q_3A_2\omega_1\omega_3 + \left(\frac{\omega^2}{2}\,q_1{}^2 + \frac{1}{2}\,\dot{q}_1{}^2\right) \times (A_2 + A_3) + \left(\frac{\omega^2}{2}\,q_2{}^2 + \frac{1}{2}\,\dot{q}_2{}^2\right)(A_1 + A_3) + \left(\frac{\omega^2}{2}\,q_3{}^2 + \frac{1}{2}\,\dot{q}_3{}^2\right)(A_1 + A_2) + \omega_1A_1(q_2\dot{q}_3 - q_3\dot{q}_2) + \omega_2A_2(q_3\dot{q}_1 - q_1\dot{q}_3) + \omega_3A_3(q_1\dot{q}_2 - q_2\dot{q}_1) \quad (14)$$

where $T_R{}^m$ is the nominal kinetic energy of the main body and $\omega^2 = \bar{\omega}\cdot\bar{\omega}$.

System Damping

Two sources of energy dissipation are represented: dissipation by discrete dampers which act about each of the n attachment points, and dissipation during flexible body deformation. Each of these is modeled as viscous friction and is introduced to the analysis by means of the concept of the dissipation function.

The viscous dissipation function for body i is given by

$$D^i = \tfrac{1}{2}\{\dot{q}^i\}^T[D_1{}^i]\{\dot{q}^i\} \quad (15)$$

where $[D_1{}^i]$ is a symmetric matrix. The first three rows and columns of $[D_1{}^i]$ correspond to the rigid body modes of body i, and since these do not involve deformation, the first three rows and columns of $[D_1{}^i]$ are zeros. The total dissipation function arising from deformation is

$$D = \sum_{i=1} D^i$$

It is assumed that discrete viscous dampers may act about any of the points at which subsidiary bodies attach to the main body. The analysis which follows proceeds from the definition of the dissipation function for the nominal motion, this function is then modified to include flexibility effects, and finally the contributions of all the discrete dampers to the equations of motion are established. The dissipation function for the dampers acting at the ith attachment point is

$$D_2{}^i = \tfrac{1}{2}\dot{\bar{\theta}}^i\cdot\tilde{\beta}^i\cdot\dot{\bar{\theta}}^i \quad (16)$$

In the case of flexible motion, $\dot{\bar{\theta}}^i$ is replaced by

$$\dot{\bar{\theta}}^i + \sum_{\alpha=1}^{3} \bar{g}_\alpha{}^i\dot{q}_\alpha{}^i + \sum_{\alpha=1}^{3} \bar{h}_\alpha{}^i\dot{q}_\alpha \quad (17)$$

where

$$\bar{g}_\alpha{}^i = \bar{\nabla} \times \bar{\phi}^i(0),\ \text{and}\ \bar{\nabla} \times \bar{\phi}_\alpha(\bar{b}^i) = -\bar{h}_\alpha{}^i$$

System Strain Energy

The system strain energy arises solely from strain energy due to appendage deformation. Attachment point springs are included as generalized forces. The strain energy function for body i is

$$U^i = \tfrac{1}{2}(q^i)^T[K^i](q^i) \quad (18)$$

where K^i is a matrix of generalized stiffnesses and q^i is a column matrix of body i flexible coordinates. The system strain energy function is

$$U = \sum_{i=1}^{n} U^i$$

Generalized Forces

Generalized forces arising from the effects of direct solar radiation pressure, gravitational field effects, and attachment point springs are included in the formulation. Other environmental or control forces and torques may be incorporated into the equations in a similar manner. The generalized forces are derived by computing the virtual work performed by the force field when the body is given a virtual displacement. The detailed derivations of the genealized forces due to solar radiation pressure, gravity and attachment point springs appear in Ref. 7, only the final results are presented below.

The generalized forces for coordinates p_α ($\alpha = 1,2,3$) are

$$Q_{HT\alpha} = \int_A \bar{P}_s dA\cdot\hat{x}_\alpha + \sum_{i=1}^{n} \int_{A^i} \bar{P}_s{}^iA^i\cdot\hat{x}_\alpha \quad (19)$$

and for main body rotational coordinates q_α ($\alpha = 1,2,3$)

$$Q_{HR\alpha} = \int_A \bar{P}_s\cdot\bar{\phi}_\alpha(\bar{s})dA + \sum_{i=1}^{n}\int_{A_i} \bar{P}_s{}^i\cdot\bar{\phi}_k{}^i(\bar{b}^i + \bar{s}^i)dA^i \quad (20)$$

and finally for body i coordinates $q_k{}^i$ ($k = 1,2,\ldots$)

$$Q_{Hk^i} = \int_{A^i}\bar{P}_s{}^i\cdot\bar{\phi}_k{}^i(\bar{s})^i dA^i \quad (21)$$

where $\bar{P}_s,\bar{P}_s{}^i$ and $\bar{\phi}_\alpha,\bar{\phi}_k{}^i$ are solar radiation pressures and modal functions for the main body and body i. The integrals extend over the sunlit surface of the spacecraft.

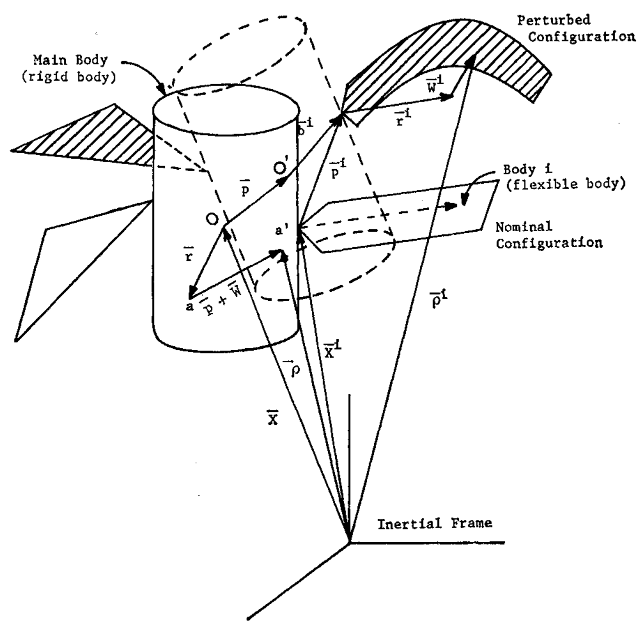

Fig. 3 Vectors defining nominal and perturbed configuration.

The generalized gravitational forces for coordinates P_α ($\alpha = 1,2,3$) are

$$Q_{gT\alpha} = -\frac{\gamma m}{(X)^3}\bar{X}\cdot\hat{x}_\alpha - \gamma\sum_{i=1}^{n}\left[\frac{m^i}{|X^i|^3}(\bar{X}^i + \bar{d}^i)\cdot\hat{x}_\alpha - \frac{3(\bar{X}^i\cdot\bar{d}^i)(\bar{X}^i\cdot\hat{x}_\alpha)m^i}{|X^i|^5}\right] \quad (22)$$

and for coordinates q_α ($\alpha = 1,2,3$) these forces are given as:

$$Q_{gR\alpha} = -(\gamma/|X|^3)\int_{B_m}\bar{r}\cdot\bar{\Phi}_\alpha(\bar{r})dm + (3\gamma/|X|^5)\int_{B_m}(\bar{X}\cdot\bar{r})[\bar{X}\cdot\bar{\Phi}_\alpha(\bar{r})]dm - \gamma\sum_{i=1}^{n}\int_{B_i}\left[\frac{\bar{X}^i}{|\bar{X}^i|^3}\cdot\bar{\Phi}_\alpha(\bar{b}^i+\bar{r}^i) + \frac{\bar{d}^i\cdot\bar{\Phi}_\alpha(\bar{b}^i+\bar{r}^i)}{|X^i|^3} + \frac{3}{|X^i|^5}(\bar{X}^i\cdot\bar{d}^i)[\bar{X}^i\cdot\bar{\Phi}_\alpha(\bar{b}^i+\bar{r}^i)]\right]dm^i \quad (23)$$

and finally for coordinates $q_k{}^i$ ($k = 1,2\ldots$)

$$Q_{gk}{}^i = -(\gamma/|X^i|^3)[\bar{X}^i\cdot\bar{R}_k{}^i + C_k{}^i] + (3\gamma/|X^i|^5)\bar{X}^i\cdot(C_k{}^i\,\bar{U} - \bar{N}_k{}^i)\cdot\bar{X}^i \quad (24)$$

where γ is the Earth gravitational constant, $C_k{}^i = \int(r_1\phi_1 + r_2\phi_2 + r_3\phi_3)_k{}^i dm^i$, $\bar{U}$ is the identity dyad and the integrals extend over the bodies.

The generalized forces associated with the springs acting about the attachment point of body i to the main body correspond to the coordinates q_α, and $q_j{}^i$, ($\alpha = 1,2,3$, $j = 1,2,3$). For q_α ($\alpha = 1,2,3$) we have

$$Q_{s\alpha} = \sum_{i=1}^{n}\bar{M}^i\cdot\bar{h}_\alpha{}^i + \sum_{j=1}^{n}\sum_{j=1}^{3}q_j{}^i\bar{g}_j{}^i\cdot\bar{K}_s{}^i\cdot\bar{h}_\alpha{}^i + \sum_{i=1}^{n}\sum_{j=1}^{3}q_j\bar{h}_j{}^i\cdot\bar{K}_s{}^i\cdot\bar{h}_\alpha{}^i \quad (25)$$

and for coordinates $q_j{}^i$ ($j = 1,2,3$)

$$Q_{s\alpha}{}^i = \bar{M}^i\cdot\bar{g}_\alpha{}^i + \sum_{j=1}^{3}q_j{}^i\bar{g}_j{}^i\cdot\bar{K}_s{}^i\cdot\bar{g}_\alpha{}^i + \sum_{j=1}^{3}q_j\bar{h}_j{}^i\cdot\bar{K}_s{}^i\cdot\bar{g}_\alpha{}^i \quad (26)$$

It was shown in Ref. 7 that the equations of motion of the system are of the form

$$[A](\ddot{x}) + [B](\dot{x}) + [C](x) = (Q) \quad (27)$$

where (x) is the vector of unknowns and may be partitioned as follows:

$$\{x\} = \{q_1{}^1\cdots q_{n_1}{}^1\,\vdots\cdots\vdots\,q_1{}^i\cdots q_k{}^i\cdots q_{n_k}{}^i\,\vdots\cdots\vdots\,q_1{}^n\cdots q_{n_n}{}^n\,\vdots\,q_1q_2q_3\,\vdots\,p_1p_2p_3\,\vdots\}^T$$

The matrices A, B, C, and Q may be partitioned in a corresponding fashion. Thus, we have for matrix A

$$[A] = \left[\begin{array}{ccccc|cc} A^{11} & \cdots & 0 & & 0 & A_1{}^{1m} & A_2{}^{1m} \\ \cdot & & \cdot & & \cdot & & \\ \cdot & & \cdot & & \cdot & & \\ \cdot & & \cdot & & & & \\ 0 & \cdots & \begin{matrix}(n_i\times n_i)\\ A^{ii}\end{matrix} & & 0 & \begin{matrix}(n_i\times 3)\\ A_1{}^{im}\end{matrix} & \begin{matrix}(n_i\times 3)\\ A_2{}^{im}\end{matrix} \\ & & \cdot & & \cdot & & \\ & & \cdot & & \cdot & & \\ 0 & & 0 & & A^{nn} & A_1{}^{nm} & A_2{}^{nm} \\ \hline A_1{}^{m1} & & \begin{matrix}(3\times n^i)\\ A_1{}^{mi}\end{matrix} & & A_1{}^{mn} & \begin{matrix}(3\times 3)\\ A_{11}{}^{mm}\end{matrix} & \begin{matrix}(3\times 3)\\ A_{12}{}^{mm}\end{matrix} \\ A_2{}^{m1} & & \begin{matrix}(3\times n_i)\\ A_2{}^{mi}\end{matrix} & & A_2{}^{mn} & \begin{matrix}(3\times 3)\\ A_{21}{}^{mm}\end{matrix} & \begin{matrix}(3\times 3)\\ A_{22}{}^{mm}\end{matrix} \end{array}\right] \quad (28)$$

In defining the various submatrices of A, the lower case letter corresponding to the capital letter designating the matrix is used to denote an element. Thus, the kj element of the submatrix A^{ii} is given by $a_{kj}{}^{ii} = m_{kj}{}^i$; that is, A^{ii} is the generalized mass matrix of body i. The matrices $A_1{}^{im}$, and $A_2{}^{im}$, are submatrices of A^{im}, the matrix coefficient of q_α, and p_α in the matrix equation corresponding to body i coordinates. An element of A^{im} is defined by

$$a_{kj}{}^{im} = \bar{\Phi}_j(\bar{b}^i)\cdot\bar{R}_k{}^i \qquad j = 1,2,3$$
$$a_{kj}{}^{im} = \hat{x}_{j-3}\cdot\bar{R}_k{}^i \qquad j = 4,5,6$$

Corresponding to main body coordinates q_α we have

$$a_{kl}{}^{mi} = \sum_{j=1}^{3}(\bar{\Phi}_k(\bar{b}^i)\cdot\hat{x}_j{}^i)(\hat{x}_j{}^i\cdot\bar{R}_l{}^i)$$

for $k = 1,2,3$ and $l = 1,2,\ldots,n_i$, whereas corresponding to the coordinates p_α we have

$$a_{kl}{}^{mi} = \sum_{j=1}^{3}(\hat{x}_k\cdot\hat{x}_j{}^i)(\hat{x}_j{}^i\cdot\bar{R}_l{}^i) \qquad \text{for } k = 4,5,6 \text{ and } l = 1,2,\ldots,n_i$$

The submatrix A^{mm} is partitioned into four parts as follows:

$$a_{kj}{}^{mm} = I_{kj}\delta_{kj} + \sum_{i=1}^{n}m^i\sum_{l=1}^{3}[\bar{\Phi}_k(\bar{b}^i)\cdot\hat{x}_l{}^i][\hat{x}_l{}^i\cdot\bar{\Phi}_j\cdot(\bar{b}^i)]$$

for $k = 1,2,3$, and $j = 1,2,3$, and where δ_{kj} is the Kronecker delta.

$$a_{kj}{}^{mm} = \sum_{i=1}^{n}\sum_{l=1}^{3}m^i(\hat{x}_k\cdot\hat{x}_l{}^i)[\hat{x}_l{}^i\cdot\bar{\Phi}_j(\bar{b}^i)] \qquad \text{for } k = 4,5,6, \text{ and } j = 1,2,3,$$

and

$$a_{kj}{}^{mm} = \sum_{i=1}^{n}\sum_{l=4}^{6}m^i[\bar{\Phi}_k(\bar{b}^i)\cdot\hat{x}_{l-3}{}^i](\hat{x}_{l-3}{}^i\cdot\hat{E}_{j-3}) \qquad \text{where } k = 1,2,3, \text{ and } j = 4,5,6$$

$$a_{kj}{}^{mm} = m\delta_{kj} + \sum_{i=1}^{n}\sum_{l=4}^{6}m^i(\hat{x}_{k-3}\cdot\hat{x}_{l-3})(\hat{x}_{l-3}{}^i\cdot\hat{E}_{j-3}) \quad k,j = 4,5,6$$

Matrix B is partitioned in the same manner as matrix A. The elements of B are as follows:

$$b_{kj}{}^{ii} = 2\bar{\omega}^i\cdot\bar{Z}_{kj}{}^i + \bar{g}_k{}^i\cdot\bar{\bar{\beta}}^i\cdot\bar{g}_j{}^i$$

for k and $j \leq n_{fi}$, and where the last term was derived in Eq. (17);

$$b_{kj}{}^{ii} = 2\bar{\omega}^i\cdot\bar{Z}_{kj}{}^i + d_{kj}{}^i$$

for k or $j > n_{fi}$ and where $d_{kj}{}^i$ is an element of $[D^i]$ defined in Eq. (15);

$$b_{kj}{}^{im} = \begin{cases}2\bar{R}_k{}^i\cdot[\bar{\omega}\times\bar{\Phi}_j(\bar{b}^i)] + \bar{g}_k{}^i\cdot\bar{\bar{\beta}}^i\cdot\bar{h}_j{}^i\\ 2\bar{R}_k{}^i\cdot[\bar{\omega}\times\bar{\Phi}_j(\bar{b}^i)]\end{cases}$$

for $j = 1,2,3$; $\quad k = 1,\ldots,n_{fi}$; $\quad$ for $k > n_{fi}$

$b_{kj}{}^{im} = 0$ for $j = 4,5,6$, and all k

$$b_{kj}{}^{mi} = \sum_{l=1}^{3}[\bar{\Phi}_k(\bar{b}^i)\cdot\hat{x}_l{}^i][\hat{x}_l{}^i\cdot(\bar{\omega}^i\times\bar{R}_j{}^i)] + \bar{h}_k{}^i\cdot\bar{\bar{\beta}}^i\cdot\bar{g}_j{}^i \qquad \text{for } j = 0,\ldots,n_{fi}, \text{ and } k = 1,2,3;$$

$$b_{jk}{}^{mi} = \sum_{l=1}^{3}[\bar{\Phi}_k(\bar{b}^i)\cdot\hat{x}_l{}^i][\hat{x}_l{}^i\cdot(\bar{\omega}^i\times\bar{R}_j{}^i)] \qquad \text{for } k = 1,2,3, \text{ and } j = n_{fi}, n_{fi}+1,\ldots,n_i$$

$$b_{kj}{}^{mi} = \sum_{l=1}^{3}(\hat{x}_{k-3}\cdot\hat{x}_l{}^i)[\hat{x}_l{}^i\cdot(\bar{\omega}^i\times\bar{R}_j{}^i)] \qquad \text{for } k = 4,5,6, \text{ and } j = 1,2,\ldots,n_i$$

$$b_{kj}{}^{mm} = \sum_{l=1}^{n} \sum_{i=1}^{3} (\hat{x}_{k-3} \cdot \hat{x}_l{}^i) [2m^i \hat{x}_l{}^i \cdot \{\bar{\omega} \times \bar{\phi}_j(\bar{b}^i)\}] \quad \text{for } k = 4,5,6, \text{ and } j = 1,2,3$$

$$b_{kj}{}^{mm} = [J(2\omega_\alpha A_\alpha)]_{kj} + \sum_{i=1}^{n} \bar{h}_k{}^i \cdot \bar{\bar{\beta}}{}^i \cdot \bar{h}_j{}^i + \sum_{i=1}^{n} \sum_{l=1}^{3} [\bar{\phi}_k(\bar{b}^i) \cdot \hat{x}_l{}^i] 2m^i \hat{x}_l{}^i \cdot [\bar{\omega} \times \bar{\phi}_j(\bar{b}^i)]$$

for k and $j = 1,2,3$, and where J is an operator defined in the Nomenclature

$$b_{kj}{}^{mm} = 0 \text{ for } k = 1,2,3, \text{ and } j = 4,5,6$$

$$b_{kj}{}^{mm} = 0 \text{ for } k = 4,5,6, \text{ and } j = 4,5,6$$

Continuing in the same fashion, the elements of C are found to be:

$$c_{kj}{}^{ii} = \dot{\bar{\omega}}{}^i \cdot \bar{Z}_{kj}{}^i + \bar{\omega}^i \cdot (\bar{\omega}^i \times \bar{Z}_{jk}{}^i) - \bar{\omega}^i \cdot \bar{\bar{E}}_{kj}{}^i \cdot \bar{\omega}^i - \bar{g}_k{}^i \cdot \bar{\bar{K}}_s{}^i \cdot \bar{g}_j{}^i \quad \text{for } k \text{ and } j = 1, \ldots, n_{fi}$$

$$c_{kj}{}^{ii} = \dot{\bar{\omega}}{}^i \cdot \bar{Z}_{kj}{}^i + \bar{\omega}^i \cdot (\bar{\omega}^i + \bar{Z}_{jk}{}^i) - \bar{\omega}^i \cdot \bar{\bar{E}}_{kj}{}^i \cdot \bar{\omega}^i + K_{kj} \quad \text{for } k \text{ and } j > n_{fi}$$

$$c_{kj}{}^{ii} = c_{jk}{}^{ii} = \dot{\bar{\omega}} \cdot \bar{Z}_{jk}{}^i + \bar{\omega}^i \cdot (\bar{\omega}^i \times \bar{Z}_{jk}{}^i) - \bar{\omega}^i \cdot \bar{\bar{E}}_{kj}{}^i \cdot \bar{\omega}^i \quad \text{for } k \text{ or } j > n_{fi}$$

$$c_{kj}{}^{im} = \begin{cases} \bar{h}_j{}^i \cdot \bar{\bar{K}}_s{}^i \cdot \bar{g}_k{}^i & \text{for } k \text{ and } j = 1, \ldots, n_{fi} \\ 0 & \text{for } k > n_{fi} \end{cases}$$

$$c_{kj}{}^{im} = \{\dot{\bar{\omega}} \times \bar{\phi}_l(\bar{b}^i) + \bar{\omega} \times [\bar{\omega} \times \bar{\phi}_l(\bar{b}^i)]\} \cdot \bar{R}_k{}^i \quad \text{for } k = 1,2, \ldots, n_i, \text{ and } j = 4,5,6,\ l = j-3$$

$$c_{kj}{}^{mi} = \sum_{l=1}^{3} [\bar{\phi}_k(\bar{b}^i) \cdot \hat{x}_l{}^i][\hat{x}_l \cdot (\dot{\bar{\omega}}{}^i \times \bar{R}_j{}^i)] - \bar{g}_j{}^i \cdot \bar{\bar{K}}_s{}^i \cdot \bar{h}_k{}^i \quad \text{for } k \text{ and } j = 1,2,3$$

$$c_{kj}{}^{mi} = \sum_{l=1}^{3} (\hat{x}_{k-3} \cdot \hat{x}_l{}^i) [\hat{x}_l \cdot (\dot{\bar{\omega}} \times \bar{R}_j{}^i)] \quad \text{for } k = 4,5,6,\ j = 1,2, \ldots, n_i$$

$$c_{kj}{}^{mm} = -\sum_{i=1}^{n} \bar{h}_k{}^i \cdot \bar{\bar{K}}_s{}^i \cdot \bar{h}_j{}^i + x_{kj} + \sum_{i=1}^{n} \sum_{l=1}^{3} m^i [\bar{\phi}_k(\bar{b}^i) \cdot \hat{x}_l{}^i] \times (\hat{x}_l{}^i \cdot [\dot{\bar{\omega}} \times \bar{\phi}_j(\bar{b}^i) + \bar{\omega}\ \{\bar{\omega} \times \bar{\phi}_j(\bar{b}^i)\}])$$

for k and $j = 1,2,3$, and where x_{kj} is the kj element of matrix $[X]$ defined as

$$c_{kj}{}^{mm} = 0 \text{ for } k = 1,2, \ldots, 6, \text{ and } j = 4,5,6$$

$$c_{kj}{}^{mm} = \sum_{i=1}^{n} \sum_{l=1}^{3} (\hat{x}_{k-3} \cdot \hat{x}_l{}^i)(m^i \hat{x}_l{}^i \cdot [\dot{\bar{\omega}} \times \bar{\phi}_j(\bar{b}^i) + \bar{\omega} \times \{\bar{\omega} \times \bar{\phi}_j(\bar{b}^i)\}]) \quad \text{for } k = 4,5,6, \text{ and } j = 1,2,3$$

$$c_{kj}{}^{mm} = 0 \text{ for } k = 4,5,6 \text{ and } j = 4,5,6.$$

It remains to define the force vector on the right side of the equation

$$Q_k{}^i = \ddot{\bar{X}}{}^i \cdot \bar{R}_k{}^i + \dot{\bar{\omega}}{}^i \cdot \bar{Y}_k{}^i + \bar{\omega}^i \cdot (\bar{\omega}^i \times \bar{Y}_k{}^i) - \bar{\omega}^i \cdot \bar{\bar{N}}_k{}^i \cdot \bar{\omega}^i + \bar{\bar{M}}{}^i \cdot \bar{g}_k{}^i + Q_{HK}{}^i + Q_{gK}{}^i \quad \text{for } k = 0,1, \ldots n_{fi}$$

$$Q_k{}^i = \ddot{\bar{X}}{}^i \cdot \bar{R}_k{}^i + \dot{\bar{\omega}}{}^i \cdot \bar{Y}_k{}^i + \bar{\omega}^i \cdot (\bar{\omega}^i \times \bar{Y}_k{}^i) - \bar{\omega}^i \cdot \bar{\bar{N}}_k{}^i \cdot \bar{\omega}^i + Q_{HK}{}^i + Q_{gK}{}^i \quad \text{for } k = n_{fi}, n_{fi} + 1, \ldots, n_i$$

$$Q_\alpha{}^{mm} = (\bar{\bar{I}} \cdot \dot{\bar{\omega}}) \cdot \hat{x}_\alpha + [(\bar{\bar{I}} \cdot \bar{\omega}) \times \bar{\omega}] \cdot \hat{x}_\alpha - \sum_{i=1}^{n} \sum_{l=1}^{3} [\bar{\phi}_\alpha(\bar{b}^i) \cdot \hat{x}_l{}^i] m^i [\ddot{\bar{X}}{}^i \cdot \hat{x}_l{}^i + \dot{\bar{X}}{}^i \cdot (\bar{\omega}^i \times \hat{x}_l{}^i) + \hat{x}_l{}^i \cdot (\bar{\omega}^i \times \bar{d}^i)] + Q_{HR\alpha} + Q_{gR\alpha} + Q_{s\alpha} \quad \text{for } \alpha = 1,2,3$$

Finally,

$$Q_i{}^{mm} = -m[\ddot{\bar{X}} \cdot \hat{x}_{i-3} - \dot{\bar{X}} \cdot (\bar{\omega} \times \hat{x}_{i-3})] - \sum_{i=1}^{n} \sum_{l=1}^{3} (\hat{x}_i \cdot \hat{x}_l{}^i) [m^i (\ddot{\bar{X}}{}^i \cdot \hat{x}_l{}^i + \dot{\bar{X}}{}^i \cdot (\bar{\omega}^i \times \hat{x}_l{}^i) + \hat{x}_l{}^i \cdot (\dot{\bar{\omega}}{}^i \times \bar{d}^i)] + Q_{HTi} + Q_{gTi} \quad \text{for } i = 4,5,6.$$

Concluding Remarks

The motion of a damped, controlled flexible spacecraft undergoing arbitrarily large nominal motion, can be simulated by a system of time-varying linear differential equations representing motion relative to prescribed nominal motion. The two principal advantages of this approach are 1) numerical solutions can be obtained which exploit the linearity to reduce simulation costs for slowly time-varying coefficients, and 2) a stability analysis can be performed to provide an indication of local stability and a means of limiting the period of time to be simulated. The work is being continued to obtain "topological tree" formulations of the equations of motion, solve numerical problems associated with modal truncation, and reduce integration cost.

$$[X] = \begin{bmatrix} \omega_1{}^2(A_2 + A_3) - \omega_2{}^2A_2 - \omega_3{}^2A_3 & -A_3(\dot{\omega}_3 + \omega_1\omega_2) & A_2(\dot{\omega}_2 - \omega_1\omega_3) \\ A_3(\dot{\omega}_3 - \omega_2\omega_1) & \omega_2{}^2(A_1 + A_3) - \omega_1{}^2A_1 - \omega_3{}^2A_3 & -A_1(\dot{\omega}_1 + \omega_2\omega_3) \\ -A_2(\dot{\omega}_2 + \omega_3\omega_1) & A_1(\dot{\omega}_1 - \omega_3\omega_2) & \omega_3{}^2(A_1 + A_2) - \omega_1{}^2A_1 - \omega_2{}^2A_2 \end{bmatrix}$$

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